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UNIMODAL BEAM ELEMENTS

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Abstract—A uniform prismatic elastic element in small displacements theory can be modeled as consisting of extension, torsion and flexural elements acting in parallel. The extension and torsion elements are unimodal, whereas the flexural elements are bimodal, due to the combined action of bending moments and shear forces. This dissimilarity between the elements is the source of inconsistency between the analysis of a truss and a framed structure, especially in a force approach. It is shown that by an appropriate change of basis, one can uncouple the flexural element into a unimodal moment element which carries the average moment of the element, and a unimodal shear element which carries the shear force and related moment. As a result any framed structure can be viewed as a generalized truss and analyzed accordingly in a standard form.

It is further shown that it is possible to physically construct a moment element and a shear element, both exhibiting unimodal deformation patterns. Consequently one can replace a classical beam element by a moment element and a shear element assembled in parallel. The approach is tested numerically in the case of beams of constant height subjected to several loading conditions. Preliminary results indicate that in theory, substantial weight reductions can be obtained when designing structures composed of parallel unimodal elements.

1. INTRODUCTION

Consider a uniform, prismatic, straight element connected to nodes A and B in 3-dimensional space. It is assumed that the element cross-section exhibits two planes of symmetry and the element is loaded only at its extremities. Within the small displacements theory of structural analysis and neglecting the effect of deformations due to shearing stresses, if the six nodal displacements of each node are given arbitrary values, the element deformation pattern can be defined by the generalized strain vector **e**

$$\mathbf{e} = (\delta \quad \theta_{A1} \quad \theta_{B1} \quad \alpha \quad \theta_{A2} \quad \theta_{B2})^T \tag{1}$$

where δ is the element total elongation, θ_{A1} and θ_{B1} are the end-rotations with respect to the chord AB in the first plane of symmetry, α is the relative torsion rotation of nodes A and B, and θ_{A2} and θ_{B2} are the end-rotations with respect to the chord AB in the second plane of symmetry.

Corresponding to the generalized strain vector we have the generalized stress vector **t**

$$\mathbf{t} = (n \quad m_{A1} \quad m_{B1} \quad q \quad m_{A2} \quad m_{B2})^T, \tag{2}$$

where n is the axial load in the element, m_{A1} and m_{B1} are the end moments in plane one, q is the torsional moment and m_{A2} and m_{B2} are the end moments in plane two.

The six strain components are related to the six stress components by the linear constitutive law

$$\mathbf{t} = \mathbf{Se} \tag{3}$$

which are the following quasi uncoupled equations

$$n = \frac{EA}{L}\delta \tag{4a}$$

$$\begin{Bmatrix} m_{A_i} \\ m_{B_i} \end{Bmatrix} = \frac{EI_i}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{Bmatrix} \theta_{A_i} \\ \theta_{B_i} \end{Bmatrix} \quad i = 1, 2 \tag{4b}$$

$$q = \frac{GJ}{L}\alpha, \tag{4c}$$

where S is the element natural stiffness matrix, L is the element length, E is Young's modulus, G is the shear modulus, A is the element cross-sectional area, J is the torsional constant and I_1 and I_2 are the moments of inertia in both planes of symmetry. We notice the uncoupling of the stretching mode, twisting mode and bending modes in both planes of symmetry. Stretching and twisting are both unimodal patterns since they are described by one generalized strain (δ and α). Bending, however, is bimodal since it is defined in each plane of symmetry by two strain components (θ_{A_i} and θ_{B_i}).

The subject matter of this paper is concerned with the uncoupling of the bending pattern into two orthogonal deformations, a pure bending mode and a "pure" shear mode. This will lead to full uncoupling of all six deformation patterns of the prismatic element, which will result in a unified and also simplified approach to the analysis of framed structures. It will also be shown more specifically that the uncoupling of the moment and shear deformations may lead to improved designs of beam structures.

Before concluding this section, a last remark. Over the years of teaching structural theory I was asked by many a student, the reason why the shear force does not appear in the stress strain relations (3). The painstaking reply would invariably be that the shear force is implied in the end moments m_A and m_B. This would however not quench their curiosity — the shear force was still missing in eqn (3) even after the explanation. The present theory makes the shear force appear explictly in the stress strain relations and in the author's view, this settles the question.

2. UNCOUPLED BENDING DEFORMATIONS

Consider the stress–strain relationship of the bending deformation in one of the planes of symmetry of an element (4b). Symbolically, these equations are represented by the stress-strain relations (3) where we consider only the two equations pertinent to bending, in one plane of symmetry. An eigenvalue analysis of the bending stiffness matrix yields the characteristic eigenvalues $2(EI/L)$ and $6(EI/L)$, and the corresponding eigenmatrix

$$\Gamma = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}. \tag{5}$$

We have chosen this eigenmatrix in order to relate the eigenvectors to appropriate structural quantities, as will become clear in the following. The two eigenvectors correspond respectively, to a symmetric and antisymmetric deformation pattern.

Performing the change of basis (6a) and using the eigenmatrix (5), we obtain the transformed stress vector (6b) and stiffness matrix (6c)

$$\mathbf{e} = \Gamma\mathbf{e}' \tag{6a}$$

$$\mathbf{t}' = \Gamma^T\mathbf{t} \tag{6b}$$

$$\mathbf{S}' = \Gamma^T\mathbf{S}\Gamma, \tag{6c}$$

where tagged quantities are written in transformed coordinates.

Solving eqn (6a) and performing the matrix multiplications in eqns (6b, 6c) yields the expressions of the transformed strain and stress components and the diagonal stiffness matrix

$$\mathbf{e}' = \begin{Bmatrix}(\theta_A - \theta_B) \\ (\theta_A + \theta_B)\end{Bmatrix} = \begin{Bmatrix}\phi \\ \psi\end{Bmatrix} \tag{7a}$$

$$\mathbf{t}' = \begin{Bmatrix}(m_A - m_B)/2 \\ (m_A + m_B)/2\end{Bmatrix} = \begin{Bmatrix} m \\ sL/2\end{Bmatrix} \tag{7b}$$

$$\mathbf{S}' = \frac{EI}{L}\begin{bmatrix}1 & 0 \\ 0 & 3\end{bmatrix}. \tag{7c}$$

To gain insight into eqns (7), consider the deformation pattern of a beam element in Fig. 1a. The original generalized strains and stresses are the relative end-rotations (θ_A, θ_B) and end moments (m_A, m_B) of the beam. We know that these quantities are coupled. Through the change of basis (7) we have established that the beam deformation is in fact the sum of two orthogonal deformation patterns, a symmetric mode characterized by the generalized strain ϕ, and an antisymmetric mode characterized by the generalized strain ψ. The physical interpretation of these strains is indicated in the figure. From eqns (7b), it is straightforward that the generalized stress component in the symmetric mode is the average bending moment m of the beam, and that the stress component in the antisymmetric mode is $sL/2$ where s is the constant shear force of the beam span.

Finally, since the transformed stiffness matrix is diagonal, we obtain the following, uncoupled stress–strain relations for bending

$$m = \left(\frac{EI}{L}\right)\phi \tag{8a}$$

$$\frac{sL}{2} = \left(3\frac{EI}{L}\right)\psi. \tag{8b}$$

Focussing our attention on the moment distribution diagrams in Fig. 1 we note that the symmetric mode carries the average bending moment, without shear, and that the

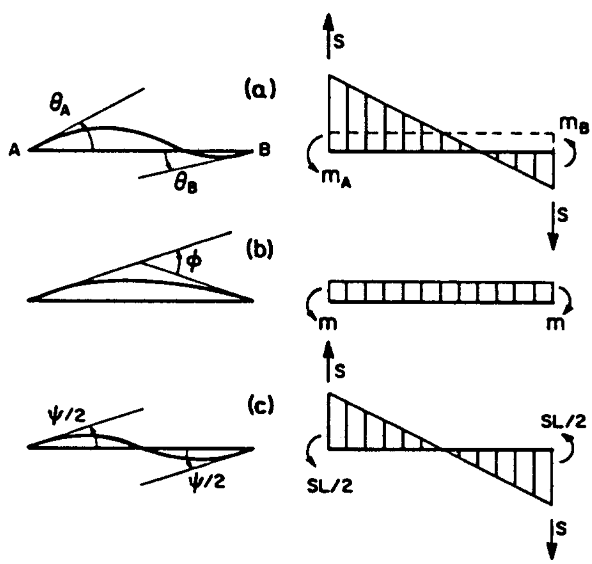

Fig. 1. Bending element deformation pattern.

antisymmetric mode carries the shear force and corresponding differential bending distribution. The sum of the two orthogonal moment diagrams is equal to the moment diagram of the beam. Similarily the sum of the two orthogonal deformation patterns is equal to the deformation of the beam element.

We have established that in essence, the bending behaviour of a beam element is composed of two orthogonal patterns. The symmetric deformation causes the beam to deform in pure moment (8a). The antisymmetric deformation (8b) causes the beam to deform in "pure" shear. Rewriting eqns (1), (2) and (4) using the new bending coordinates, we obtain the following strain, stress and diagonal stiffness components

$$\mathbf{e} = (\delta \quad \phi_1 \quad \psi_1 \quad \alpha \quad \phi_2 \quad \psi_2)^{\mathrm{T}} \tag{9a}$$

$$\mathbf{t} = \left(n \quad m_1 \quad \frac{s_1 L}{2} \quad q \quad m_2 \quad \frac{s_2 L}{2}\right)^{\mathrm{T}} \tag{9b}$$

$$\mathbf{k} = \left(\frac{EA}{L}\ \frac{EI_1}{L}\ \frac{3EI_1}{L}\ \frac{GJ}{L}\ \frac{EI_2}{L}\ \frac{3EI_2}{L}\right)^{\mathrm{T}}. \tag{9c}$$

These are the six orthogonal deformation patterns of the beam element: extension and torsion, and moment and shear in both planes of symmetry.

3. A UNIFIED THEORY OF STRUCTURAL ANALYSIS

In the classical theory of prismatic structures it is recognized that a member can be visualized as being composed of four generic elements, attached in parallel to the element nodes and which deform in orthogonal patterns: extension, torsion and bending in orthogonal planes of symmetry. The stretching and torsion deformations are unimodal and can be treated in a similar manner. Bending, however, is different. It is influenced by two coupled quantities, the bending moment and the shear force.

The coupling of the bending moment and the shear force in beam elements is the source of some dissimilarity between the analysis of redundant trusses and the analysis of redundant beam structures when using a force method. A statically redundant truss can usually be rendered determinate by removing a suitable subset of superfluous bars. In the case of beams however, as shown for instance in McGuire and Gallagher (1979), one introduces "cuts" in the structure releasing either stress resultants or support reactions. The basic structure here is not a subset of the original structure but rather the original structure, cut open at distinct locations. The difference is not merely semantic. One has to include the unknown support reactions in the equilibrium equations, a requirement which is not necessary in a truss-type analysis. This is the source of a score of algorithmic complications in a computerized implementation of the process. In the present approach, and after the change of coordinates at the beam element level, we can formulate a unified theory of structural analysis where all the generic elements are treated in exactly the same manner.

A straight prismatic member can be modeled as six unimodal, orthogonal elements connected in parallel. Each element has its own natural stiffness k_i (9c). If the end nodes are given small arbitrary displacements, each element deforms in its natural strain mode e_i (9a). The end loads acting on each element are $t_i = k_i e_i$. The strain energy in the member is the sum of the strain energies of the six elements

$$U = \frac{1}{2} \sum_{i=1}^{6} k_i e_i^2. \tag{10}$$

The analysis of a prismatic structure is now, in all respects, similar to the analysis of a generalized truss where each member is decomposed into six parallel elements. Let **u** and **p** be the N-vectors of unconstrained nodal displacements and nodal loads, respectively, in global coordinates. Let S be the unassembled diagonal matrix of the stiffness of the elements of the structure (size = $6M$ where M is the number of members). and let **t** and **e** be the corresponding generalized stresses and deformations of all the elements. The basic equations for analyzing the structure are

$$\begin{aligned} \mathbf{Qt} &= \mathbf{p} \quad \text{Equilibrium} \\ \mathbf{Ru} &= \mathbf{e} \quad \text{Strain–displacement} \\ \mathbf{Se} &= \mathbf{t} \quad \text{Constitutive law,} \end{aligned} \tag{11}$$

where **Q** is the statics matrix and $\mathbf{R} = \mathbf{Q}^T$ is the kinematics matrix. Simple substitution leads to the equilibrium equations of the Displacement method

$$\mathbf{Ku} = \mathbf{p}; \quad \mathbf{K} = \mathbf{QSR}. \tag{12}$$

The elegance of the present approach emerges even more in the Force method. The degree of statical redundancy of the structure is $(6M - N)$. The structure can be made determinate by (physically) removing $(6M - N)$ redundant elements. This corresponds to finding an $N \times N$ non-singular submatrix in **Q**. Any simplex type algorithm can be employed for that purpose. The eliminated elements may include axial, moment, shear or torsion elements from any of the members of the structures. Matrices **Q** and **S** and vectors **e** and **t** are subdivided accordingly. Using subscript b for the basic structure and subscript r for the redundant elements, eqns (12) can be rewritten as detailed in Fuchs (1982)

$$\mathbf{Q}_b\mathbf{t}_b + \mathbf{Q}_r\mathbf{t}_r = \mathbf{p} \tag{13a}$$

$$\mathbf{R}_b\mathbf{u} = \mathbf{e}_b; \quad \mathbf{R}_r\mathbf{u} = \mathbf{e}_r \tag{13b}$$

$$\mathbf{S}_b\mathbf{e}_b = \mathbf{t}_b; \quad \mathbf{S}_r\mathbf{e}_r = \mathbf{t}_r. \tag{13c}$$

Compatibility in terms of element strains is obtained by eliminating the nodal displacements from eqns (13b)

$$\mathbf{e}_r = \mathbf{R}_r\mathbf{R}_b^{-1}\mathbf{e}_b, \tag{14}$$

which after substitution yields the compatibility equations in terms of element stresses

$$(\mathbf{F}_r + \mathbf{R}_r\mathbf{R}_b^{-1}\mathbf{F}_b\mathbf{Q}_b^{-1}\mathbf{Q}_r)\mathbf{t}_r = \mathbf{R}_r\mathbf{R}_b^{-1}\mathbf{F}_b\mathbf{Q}_b^{-1}\mathbf{p} \tag{15}$$

where the **F**s are the diagonal matrices of element flexibilities.

To complete the picture, consider the computation of a nodal displacement by the unit load method

$$u_j = \sum_{m=1}^{M} \sum_{i=1}^{6} \frac{1}{k_{im}} t_{im} t_{im}^{(j)} \tag{16}$$

where subscript m is the member number, subscript i is the unimodal element number and the $t_{im}^{(j)}$ are element stresses due to a unit load in the j direction. The summation includes terms of the type

$$\frac{1}{12}\left(\frac{L^3}{EI}\right)_m s_m s_m^{(j)}, \tag{17}$$

which are the contributions of the shear forces to the total deformation of the structure. Shear deformation is thus given its proper place in structural analysis. These deformations originate from the normal stresses due to shear and cannot be neglected. The deformations derived from the tangential stresses due to shear are those which are usually omitted.

4. UNIMODAL BEAM ELEMENTS

The main result obtained so far, is that a prismatic beam element (bimodal element) is mathematically equivalent to two unimodal elements mounted in parallel, a moment element and a shear element. When the nodes are given symmetric rotations [$\theta_B = -\theta_A$ in eqn (7a)] the moment element is strained and reacts in pure bending with a stiffness given by eqn (8a). The shear element remains unstrained ($\psi = 0$) and consequently has zero stiffness in a symmetric deformation pattern. Similarly, if the nodes rotate anti-symmetrically [$\theta_A = \theta_B$ in eqn (7a)] the shear element is strained and reacts in pure shear, with a stiffness given by eqn (8b). The moment element is not strained ($\phi = 0$), i.e. it has zero stiffness in an anti-symmetric pattern. If the nodes are given arbitrary rotations, the moment element takes the symmetric part of the deformation and the shear element takes the anti-symmetric part of the deformation.

A question which comes to mind is whether the moment element and the shear element can be given a physical reality. In other words, can we construct a moment beam and a shear beam. In the affirmative, a following question would be whether there is any advantage in using a moment beam and a shear beam mounted in parallel instead of a regular bimodal beam. We will address the first of these two questions in this section.

Consider a straight uniform beam with a frictionless, normal guide at mid-span (Fig. 2a, left). The stress–strain relations of this element are

$$\begin{Bmatrix} m_A \\ m_B \end{Bmatrix} = \frac{EI}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_A \\ \theta_B \end{Bmatrix}. \tag{18}$$

Applying the change of basis (5)–(7) to these equations one obtains the modal equilibrium equations

$$\begin{Bmatrix} m \\ \frac{sL}{2} \end{Bmatrix} = \frac{EI}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \phi \\ \psi \end{Bmatrix}. \tag{19}$$

These are exactly the equations of a moment element. This element has a stiffness of EI/L in pure bending, and no stiffness in pure shear. Figures 2b and 2c show clearly that this beam has a deformation in pure bending, but has a rigid-body displacement in pure shear.

Consider now a uniform beam with a frictionless hinge at mid-span (Fig. 2a, right). The stress–strain relations are in this case

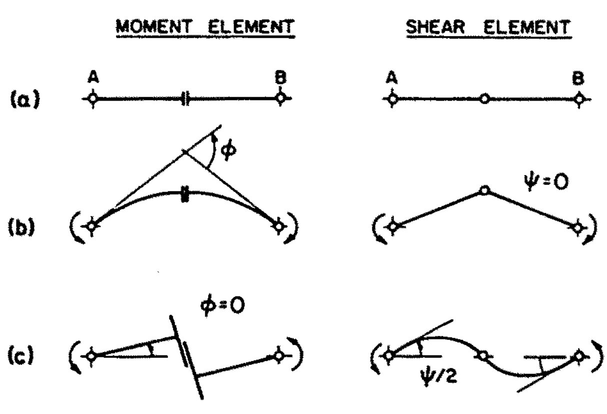

Fig. 2. Moment element and shear element.

$$\begin{Bmatrix} m_A \\ m_B \end{Bmatrix} = \frac{EI}{L}\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{Bmatrix} \theta_A \\ \theta_B \end{Bmatrix}. \tag{20}$$

which after the same change of basis, produces the modal equilibrium equations

$$\begin{Bmatrix} m \\ \dfrac{sL}{2} \end{Bmatrix} = \frac{EI}{L}\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{Bmatrix} \phi \\ \psi \end{Bmatrix}. \tag{21}$$

As expected these are the equations of a shear element. The element has a stiffness $3EI/L$ in pure shear and no stiffness in pure bending. As can be seen in Figs 2b and 2c, the beam deforms only when in pure shear but has a rigid-body displacement in pure bending.

If we add the load–deformation equations of the moment element and the shear element [eqns (18) and (20)], we obtain the relations of the uniform beam (4b). All this leads to the conclusion that one can actually replace a uniform beam by a moment beam and a shear beam (with same cross-sectional properties) mounted in parallel. The boundary conditions of both beams are such that they must rotate with the same angles, θ_A and θ_B. The moment beam will carry the average moment m and the shear beam will carry the shear s and related differential moments.

The displacement of each beam is, in general, composed of its natural deformation pattern plus a rigid-body displacement for which the beam presents no stiffness. Consider a typical assembly of a moment beam and a shear beam (Fig. 3). When subjected to arbitrary end displacements (δ_7–δ_{10}), the rigid-body motions result in a slope discontinuity $(\delta_2-\delta_3)$ at mid-span in the shear beam and a transverse displacement discontinuity $(\delta_5-\delta_6)$ at mid-span in the moment beam, given by

$$\begin{Bmatrix} \delta_2-\delta_3 \\ \delta_5-\delta_6 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \delta_7 \\ \delta_8 \\ \delta_9 \\ \delta_{10} \end{Bmatrix}. \tag{22}$$

A typical displacement pattern of a cantilever composed of parallel unimodal beams is also shown in Fig. 3. The tip displacement is identical to what would have been obtained with a regular (bimodal) uniform beam of the same stiffness, EI.

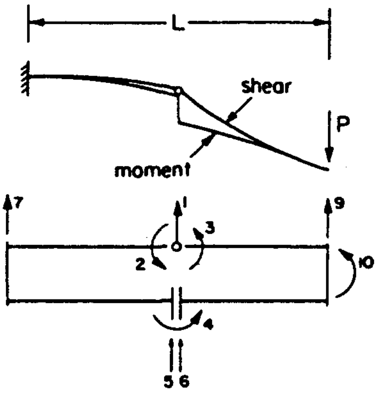

Fig. 3. Tip-loaded unimodal cantilever beam.

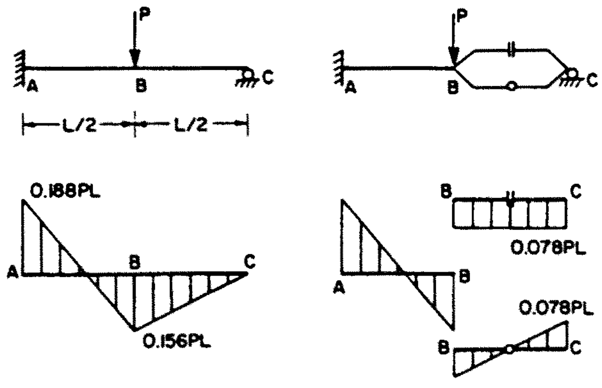

Fig. 4. Analysis of a propped cantilever.

5. DESIGNING WITH UNIMODAL BEAM ELEMENTS

Consider the example of a propped, cantilever beam of uniform stiffness EI, span L and subjected to a concentrated force P at mid-span (Fig. 4). If we replace section BC of the structure by a pair of unimodal beams with same EI, we emulate the structural response of the original beam. The nodal displacements at B and C and the bending moment diagram in section AB are the same. The bending moment in section BC is split between the moment beam and the shear beam. At first glance there seems to be no obvious reason to replace the original section BC by a pair of unimodal beams, since by doing so, one merely doubles the weight of that section.

However, one will notice that each one of the unimodal beams is significantly over-designed when compared to the original section, since the maximum bending moment in the unimodal beams is half the maximum bending moment in the original beam (0.078 compared to 0.156). One should bear in mind that the unimodal beams have the same EI as the original beam. There is therefore room to reduce the stiffnesses of the unimodal beams and increase their stress levels. Thus by using unimodal beams we design a more flexible structure while maintaining the weight of the original structure.

Further weight reduction can be obtained from the following argument. Since the moment and shear beams are two separate entities, there is no need to design both beams with identical cross-sections. One can very well conceive designs with different EIs for the moment beam and the shear beam. In fact, we will see in the following that the introduction of two independent design variables for each unimodal beam is cardinal for the design of efficient unimodal beam structures.

For this purpose, consider the design of the propped cantilever, loaded at the mid-span (Fig. 5). We will assume that the beam elements are of constant height h and that the weight of a beam is proportional to the cross-sectional area of the flanges. In the case of the classical bimodal beams (Fig. 5a), the design variables are the moments of inertia x_1 and x_2 of sections AB and BC, respectively. The objective function is the volume of the elements (linear in the design variables) and the constraints of the problem are the maximum stresses in each element. The non-dimensional design variables x_i and the non-dimensional allowable stress β, are defined as

$$x_i = I_i/h^4 \quad i = 1,2$$
$$\bar{\sigma} = \beta(PL/h^3), \tag{23}$$

where $\bar{\sigma}$ is the allowable stress (in tension and compression).

The optimum design values and the minimum volume are:

$$x_1^* = 0.100/\beta; \quad x_2^* = 0.075/\beta$$
$$v^* = 0.35Lh^2/\beta. \tag{24}$$

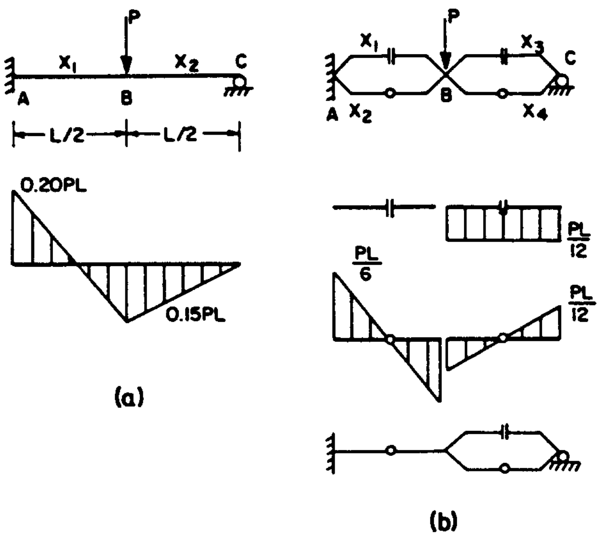

Fig. 5. Optimum design of a propped cantilever.

At the minimum point the stresses σ_A and σ_{BC} reach the limiting stress value. The bending moment diagram for the optimum design is given in Fig. 5a. Note, the minimum volume of the two-variable, propped cantilever is 6.7% lower than the minimum volume in the case of a uniform cross-section along the entire beam, for which we have $v^* = 0.375Lh^2/\beta$.

Focusing our attention now on the optimum design of a propped cantilever composed of two pairs of unimodal beams (Fig. 5b), we note that we are in the presence of a minimization problem with four design variables x_i ($i = 1, 4$), where the first two variables are the non-dimensional moments of inertia of the moment and shear beams of section AB, and the latter two variables are the corresponding moments of inertia of section BC. The optimum design values for the unimodal arrangement are

$$x_1^* = 0.0; \quad x_3^* = 1/24\beta$$
$$x_2^* = 1/12\beta; \quad x_4^* = 1/24\beta$$
$$v^* = 0.333Lh^2/\beta. \tag{25}$$

It is important to emphasize at this point three properties of the design: (1) the optimum value is lower than the optimum design obtained using bimodal elements [eqn (24)] which is an encouraging result; (2) the moment beam of section AB has vanished from the design thus yielding the structure shown at the bottom of Fig. 5b; (3) the maximum stress in each beam is equal to the limiting stress. In other words the optimum structure is statically determinate (property 2) and fully stressed (property 3).

The methodology derived in this work allows us to treat the beam problem as a generalized truss. The propped cantilever has one degree of static redundancy since it is composed of four elements (four static unknown) for which we can write only three nodal equilibrium equations. Since it is subjected to one loading condition only, we obtain the well-established rule in truss design that under a single loading condition, the optimum structure is statically determinate and fully stressed (see Reinschmidt *et al.*, 1966). In fact the optimum design can be reached very easily using the stress-ratio method.

Such a result could not have been obtained without the present theory. The bimodal mathematical programming formulation can reach a statically determinate solution only by driving section BC to zero, thus producing the simple cantilever AB. However the optimum weight of this structure, $0.5Lh^2/\beta$, is larger than the result in eqn (24).

6. THE BUTTERFLY SHEAR BEAM

What has been established up to this point leads us to the tentative conclusion that improved results could be expected when designing beam structures with unimodal sections. It will be argued that the small reduction in volume obtained in the unimodal, propped cantilever example does not warrant the use of such elements. The purpose of this section is to show that the employment of "butterfly" shear beams instead of uniform shear beams, yields substantial additional reduction of the volume of the structure.

A common rule of thumb to reduce structural weight is to increase the stress levels in the structure as much as possible without violating the constraints. In other words, the implied purpose is to design structures which are in some sense "as close as possible" to being fully stressed. Herein lies the intrinsic efficiency of the truss element as compared to the bimodal beam element. The truss element is uniformly stressed. The stress constraint, if active in a truss element, is active all over the length of the element. The bimodal beam element however is usually active only on a particular section of the element. The other cross-sections of the element experience lower stress levels. By employing unimodal elements we have come nearer to a state of uniform stress. The moment beam is uniformly stressed. The shear beam however still has linearly varying stress levels with extremum values at the end-sections only. It is to be expected that a shear beam with uniform stresses will therefore furnish better structural results.

To this respect consider the beam element drawn in Fig. 6. This is an element of constant height with "butterfly" shaped flanges. The flanges are linearily tapered with zero width in the middle of the element (at the hinge location) and maximum width at the element ends. Neglecting the contribution of the web to the bending stiffness of the element, the butterfly shear element has the following stress–strain relation

$$\frac{sL}{2} = \left(2\frac{EI}{L}\right)\psi \tag{26}$$

where I is the moment of inertia at the beam ends. The stiffness of the butterfly beam is thus 2/3 the stiffness of the uniform shear beam with the same EI [eqn (8b)]. The volume of the butterfly beam is half the volume of the corresponding uniform shear beam. Since the ratio of the bending moment over the section modulus is constant along the element, the butterfly beam is a uniformly stressed element.

The effect of employing butterfly beams is immediate. If we replace the uniform shear beams by butterfly beams in the example of the propped cantilever (Fig. 5b), the optimum value becomes

$$v^* = 0.2085Lh^2/\beta \tag{27}$$

which is an improvement of 40.5% over the volume of the optimum bimodal design ($0.35Lh^2/\beta$). One will recall that the optimum design with uniform shear beams gave only a 4.8% volume reduction. The low volume (27) is a direct consequence of the fact that the

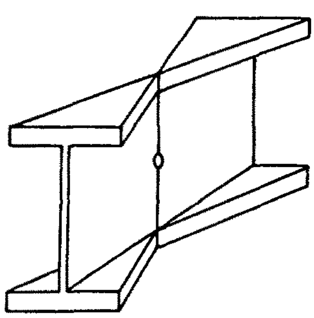

Fig. 6. The butterfly shear beam.

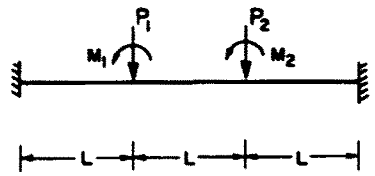

Fig. 7. A fixed–fixed beam example.

optimum design is now literally, fully stressed. Similar results were obtained in further examples, some of which are reported in the next section.

7. NUMERICAL RESULTS

The effectiveness of employing unimodal elements in beam design problems was tested more thoroughly in the case of a beam fixed at both ends and loaded at one third of the span, measured from the extremities, by concentrated forces P_1, P_2 and by couples M_1, M_2 (Fig. 7). Twenty combinations of loadings were considerd by arbitrarily assigning values one or zero to the forces and to the couples (Table 1). Six loadings were single loading conditions and 14 loadings were double loading conditions. The structure was designed for minimum volume and subjected to stress constraints, equal in tension and compression. The beam elements are of constant depth and the design variables are the moments of inertia in sections AB, BC and CD of the beam.

Four element types were compared: bimodal uniform, bimodal tapered, unimodal uniform and unimodal butterfly. The bimodal uniform element is the classical uniform beam. The three design variables of the problem are the cross-sectional areas of the flanges of the three elements. The bimodal tapered element has a flange which varies linearily between the two ends of the element. This case presents four design variables: the cross-sectional areas at sections A, B, C and D. The unimodal uniform structure is composed of six uniform unimodal elements, a unimodal moment and a unimodal shear element for each section. This structure thus has six design variables. The unimodal butterfly structure is similar to the previous one, except for the shear elements which are now butterfly shear elements. This design type has also six design variables.

Table 1. Comparison of optima of the fixed–fixed beam

Loading									Bimodal		Unimodal	
	First				Second				Uniform	Tapered	Uniform	Butterfly
Case	P_1	P_2	M_1	M_2	P_1	P_2	M_1	M_2	Type 1	Type 2	Type 3	Type 4
1	1	0	0	0	0	0	0	0	100	62	92	59
2	1	1	0	0	0	0	0	0	100	63	80	54
3	1	0	1	0	0	0	0	0	100	59	111	68
4	0	1	0	1	0	0	0	0	100	65	93	68
5	0	0	1	0	0	0	0	0	100	65	99	69
6	1	0	0	1	0	0	0	0	100	77	86	63
7	1	0	0	0	0	0	1	0	100	78	107	74
8	1	0	0	0	0	1	0	0	100	81	90	55
9	1	0	0	0	0	0	0	0	100	78	97	66
10	0	1	0	0	0	0	0	1	100	78	107	73
11	0	0	1	0	0	0	0	1	100	90	89	63
12	1	1	0	0	0	0	1	0	100	78	91	59
13	0	1	1	0	1	0	0	0	100	84	95	58
14	0	1	1	0	0	0	0	1	100	81	103	68
15	0	0	1	1	1	0	0	0	100	88	124	64
16	1	0	1	0	0	1	0	0	100	75	90	53
17	1	0	1	0	0	0	0	1	100	78	92	60
18	1	1	1	0	0	0	0	1	100	82	99	63
19	1	1	0	1	0	0	1	0	100	74	88	63
20	1	0	1	1	0	1	0	0	100	87	116	58

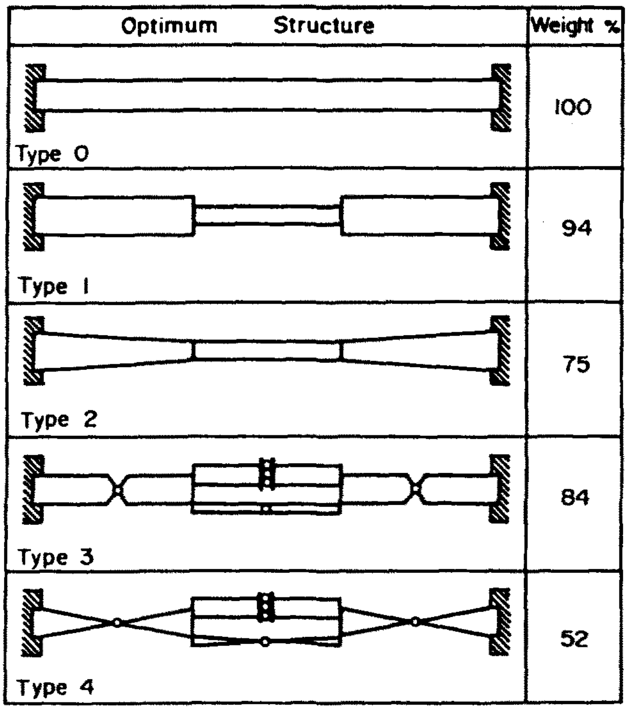

Fig. 8. Optimal layouts of a fixed-fixed beam.

The bimodal tapered element was introduced in order to provide a fair competition to the unimodal butterfly shear element. The stiffness matrix of this element was derived using a finite element approach, with Hermite interpolation functions for the displacements. The minimization was performued using the FSD iterative method (Reinschmidt *et al.*, 1966). We thus compare near optimum results; however, it is a reasonable assumption that the established trends are probably not affected by this approximation.

Table 1 gives a summary of the results. Additional information can be found in Benyamini and Bousso (1989). The optimum values were, for each case, normalized with respect to the weight of the uniform bimodal case. Comparing the uniform bimodal beams with the uniform unimodal beams (types one and three), we notice that the improvement is not very significant and in many cases, even negative. The unimodal butterfly elements (type four) yield a dramatic weight reduction when compared to the uniform bimodal elements (type one).

The optimum configurations are, as expected, shared between the tapered bimodal elements (type two) and the butterfly unimodal elements (type four), with a 17/3 rating in favor of the latter. The unimodal butterfly elements either yield about the same weights as the tapered elements or produce improved results as in case eight (Fig. 8). Note, the structure has, in this case, shed the moment elements in spans AB and CD thus producing a statically determinate structure. This feature was established in many cases, which leads us to a preliminary finding that the use of unimodal elements is especially interesting in the case where a statically determinate optimum can be obtained.

8. CONCLUSIONS

The main purpose of this paper was to unify the analysis of framed structures by modifying the treatment of the bending elements. A suitable change of basis allowed us to uncouple the pure bending mechanism and the "pure" shear mechanism of a beam element. Consequently a straight prismatic element in 3-dimensional space became structurally equivalent to six unimodal elements mounted in parallel. As a result, a framed structure can now be viewed as a generalized truss and analyzed accordingly in a unified manner. In addition, the shear force appears explicitly in the analysis equations thus giving it a proper representation along with the axial, torsion and moment loads.

It was also shown that the uncoupling of the moment and shear modes could lead to improved designs of beam structures through the use of unimodal moment and shear elements instead of the classical beam element. Preliminary results seem to indicate that significant weight reduction can be expected when the moment element is used in conjunction with a butterfly shear element. Such an assembly annuls, in fact, the inherent inefficiency of a beam element since the unimodal elements are uniformly stressed. The fact that the unimodals are not significantly better than the tapered elements is, in the author's opinion, not very relevant from an engineering viewpoint. Tapered elements have to be machined on order. The unimodals on the contrary can be factory produced and shipped with standard cross-sections thus allowing the designer to select appropriate beams from a catalogue, much in the same way as is done nowadays for bimodal beams.

In practice, the unimodal elements must be considered with some care. They require frictionless mechanisms at mid-span and the end sections of the elements must be interconnected in a rigid manner in order to have identical lateral displacements and rotations. This is not a trivial task. In addition, the theory does not as yet tackle lateral stability considerations, and local flange and web buckling. However, the potential weight reduction obtained through the unimodal elements does warrant further investigation.

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